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Exercises for 'Topics in complex analysis'

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H 4.1 (The construction of Mittag-Leffler)
Construct explicitly (i.e. as an explicit series) a holomorphic function $f : \mathbb{C}\setminus\{\sqrt{n} : n \in \mathbb{N}\} \to \mathbb{C}$ such that at $\sqrt{n}$ the function f has the principal part $q_n(z) = \frac{\sqrt{n}}{z-\sqrt{n}}$.
Hint: Prove that a second order Taylor polynomial p_n of q_n yields the local normal convergence of the sum $f = \sum_{n\in\mathbb{N}} q_n - p_n$ (cf. the statement of Theorem 2.2).

H 4.2 (The partial fraction decomposition of $\frac{\pi^2}{\sin^2(\pi z)}$)
The goal of this exercise is to prove the formula

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}.$$

This will be achieved in several steps. Keep in mind that $\sin(z) = \frac{1}{2i}\left(\exp(iz) - \exp(-iz)\right)$.

a) Show that the function $f(z) := \frac{\pi^2}{\sin^2(\pi z)}$ has its singularities exactly at the points $z^* \in \mathbb{Z}$ and determine the principal parts of the Laurent series expansion in those points.

b) Show that the series $g(z) := \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$ converges locally uniformly on $\mathbb{C}\setminus\mathbb{Z}$, so that it is meromorphic. Conclude that the difference $g(z) - f(z)$ can be extended to an entire function.

c) Show that the function $z = x + iy \mapsto f(x + iy)$ vanishes when $|y| \to +\infty$ uniformly in x.

d) Show the same statement for the function g. Then prove that the difference $g - f$ is bounded on $\mathbb{C}$ and conclude the proof.
Hint: For d) it can be useful to note that $(g - f)(z + 1) = (g - f)(z)$ for all $z \in \mathbb{C}$.

H 4.3 (The Weierstrass elliptic function)
Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. Show that up to an additive constant, there exists exactly one holomorphic function $\wp : \mathbb{C}\setminus\{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\} \to \mathbb{C}$ such that

(i) $\wp$ has principal part $q(z) = \frac{1}{z^2}$ at $d = 0$;

(ii) $\wp$ is $\{\omega_1, \omega_2\}$-periodic, i.e.

$$\wp(z+\omega_1) = \wp(z) \quad \text{and} \quad \wp(z+\omega_2) = \wp(z) \qquad \forall z \in \mathbb{C}\setminus\{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

You can use without proof that $\displaystyle\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m^2+n^2)^{3/2}} < +\infty.$

Remark: If we require that the coefficient a_0 of the Laurent series at the origin vanishes, this function is called the Weierstrass $\wp$ function. This function $\wp$ and its derivative can be used to parametrize elliptic curves. The Laurent series coefficients of $\wp$ are called Eisenstein series, and are the simplest examples of modular forms.